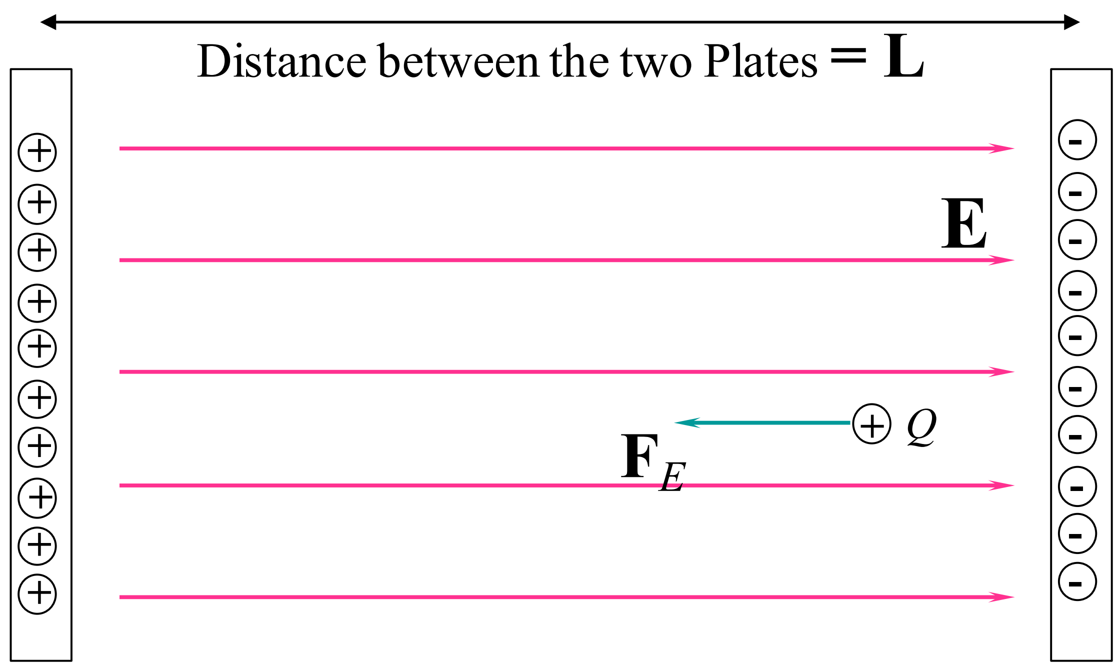
Engineering Electromagnetics

Chapter 4:

Energy and Potential

Point Charge in an External Field

To move charge Q against the electric field, a <u>force</u> must be applied that <u>*counteracts*</u> the force **F** on Q that arises from the field **E**:
(<u>i.e the charge Q has to **gain some sort of energy** (Force or work to be done) to move it against **E**</u>)



This is the reference side.

Source Charges

Sink Charges

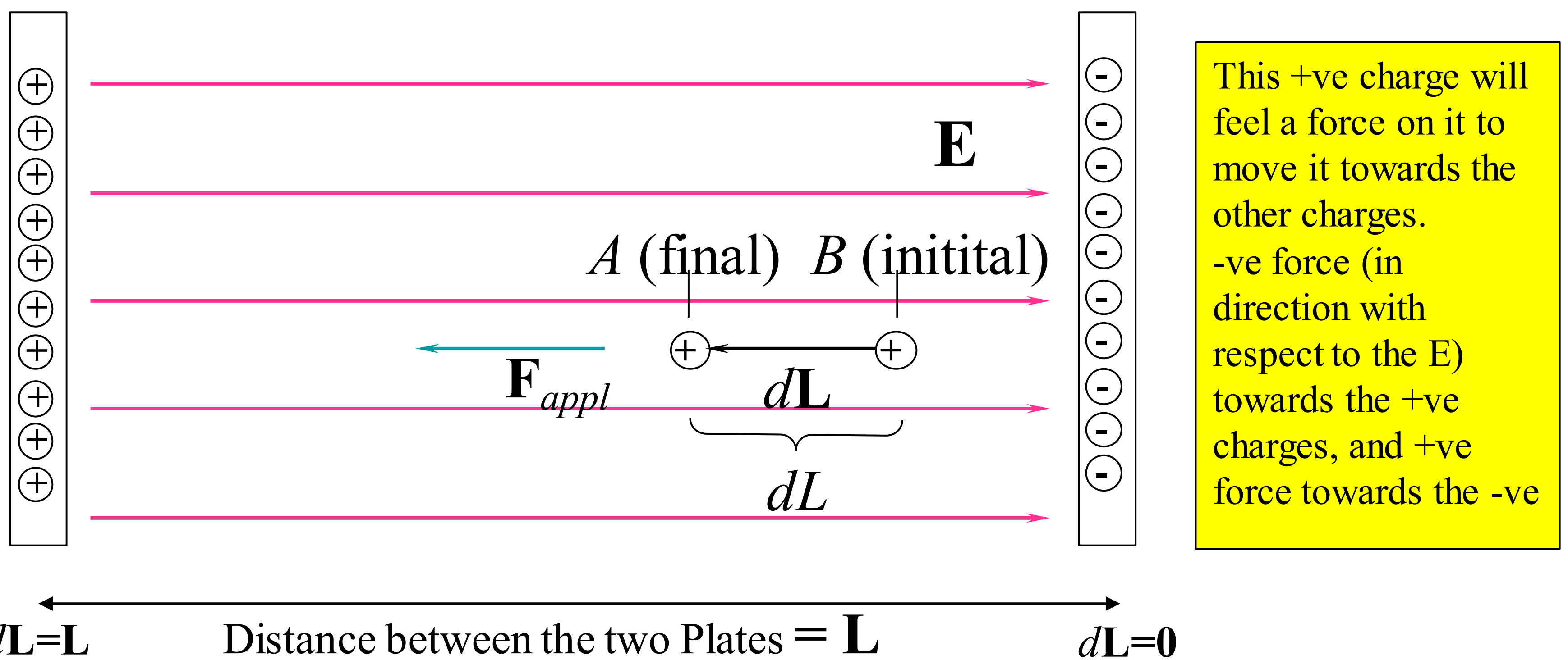
$$\mathbf{F}_E = -Q\,\mathbf{E}$$

Differential Work (Energy) Done on Moving a Point Charge Against an External Field

In moving point charge *Q* from initial position *B* over a differential distance *dL* (to final position *A)*, the differential work expended is:

$$dW = F_{appl}\, dL = -QE\, dL = -Q\mathbf{E}\cdot d\mathbf{L} \text{ [J]}$$

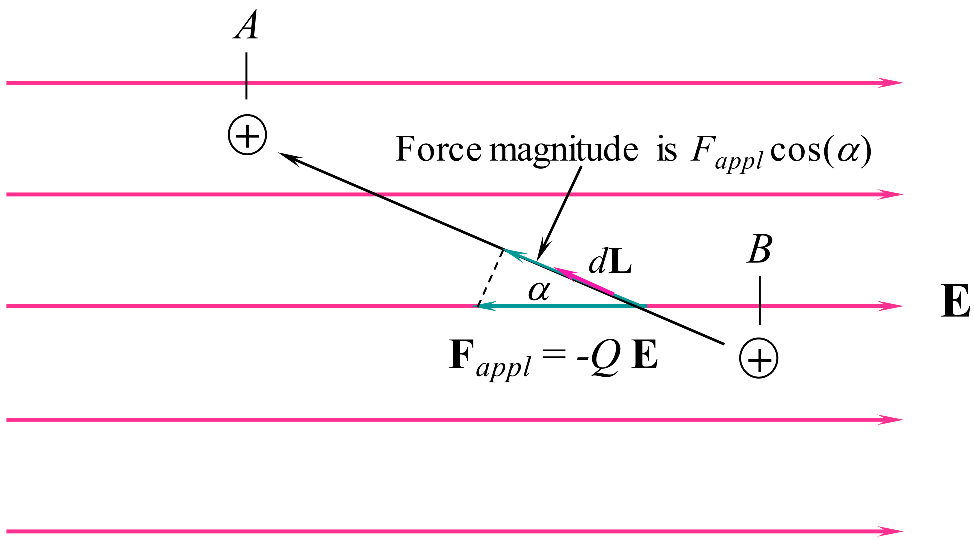
gives positive result if charge is forced *against* the electric field



The path is along an electric field line (in the opposite direction), and over the differential path length, the field can be assumed constant.

Forcing a Charge Against the Field in an Arbitrary Direction

What matters now is the component of force in the direction of motion.



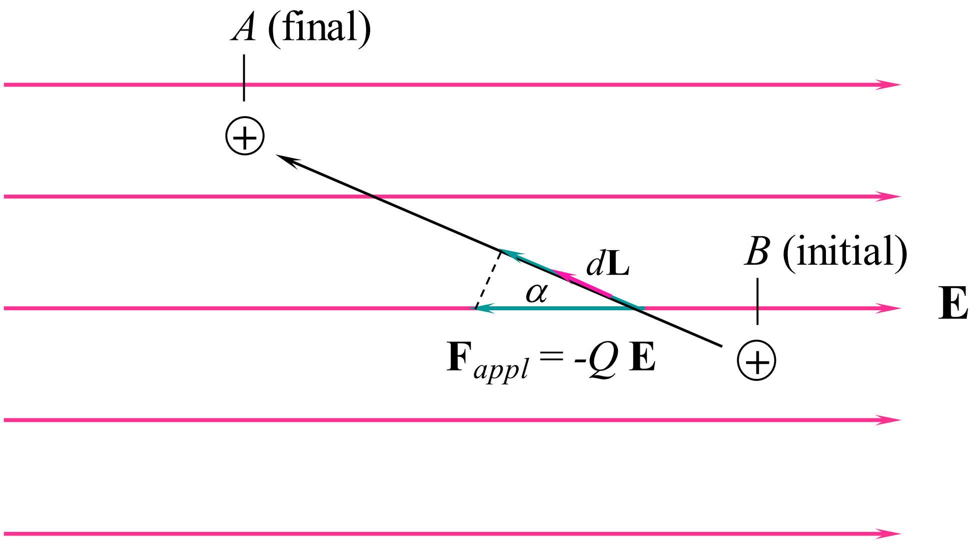
Differential work in moving charge Q through distance dL will be:

$$dW = F_{appl}\cos(\alpha)\, dL = -Q\mathbf{E} \cdot d\mathbf{L}$$

Total Work Done (i.e. Total Energy expenditure)

All differential work contributions along the path are summed to give:

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$



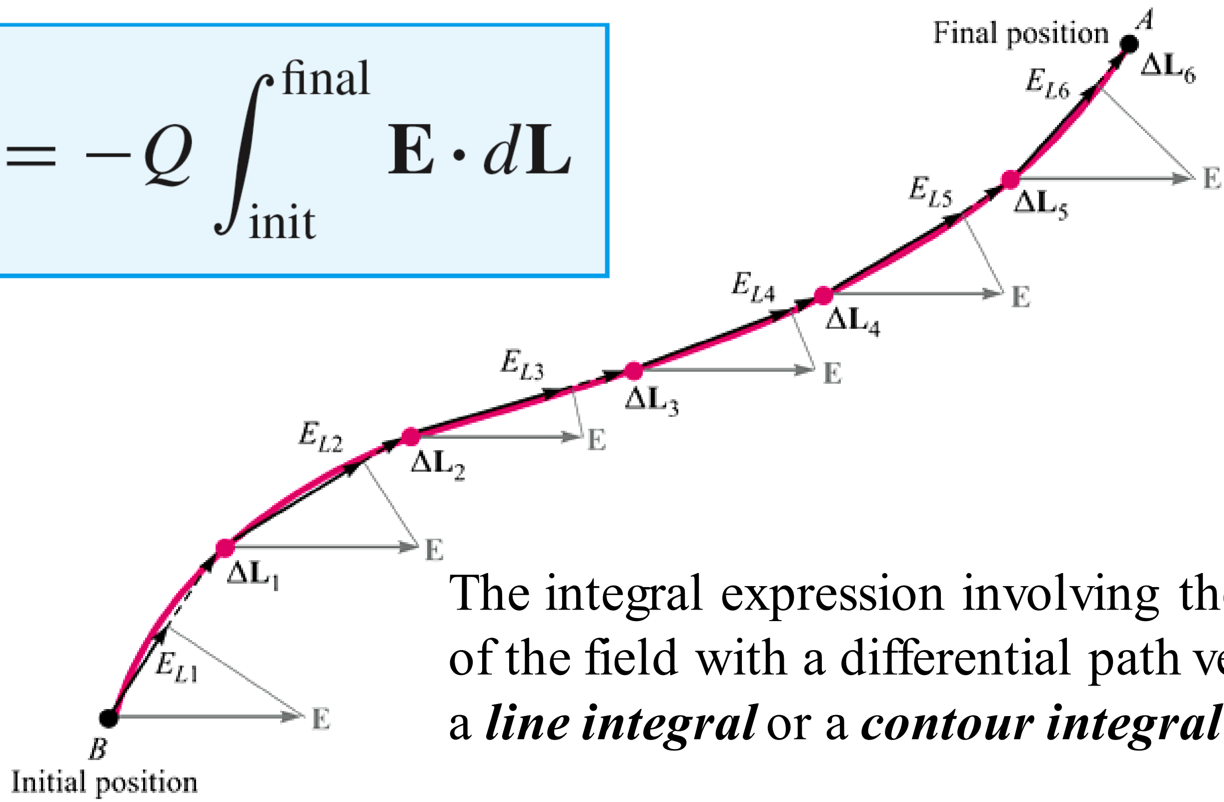
Note that the path must be specified before the integral can be evaluated., and the charge is assumed to be at rest at both its initial and final positions.

Total Work Done over an Arbitrary Path

The integral expression for work is completely general: Any shape path may be taken, with the component of force evaluated on each differential path segment.

1. Choose a path,
2. break it up into a large number of very small segments,
3. multiply the component of the field along each segment by the length of the segment, and
4. then add the results for all the segments.

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$



The integral expression involving the scalar product of the field with a differential path vector is called a ***line integral*** or a ***contour integral***.

Line Integral Evaluation

We wish to find: $$\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

where $$\mathbf{E} = E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z$$

and $$d\mathbf{L} = dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z$$

using these:

$$\int_B^A \mathbf{E} \cdot d\mathbf{L} = \int_B^A (E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z) \cdot (dx\, \mathbf{a}_x + dy\, \mathbf{a}_y + dz\, \mathbf{a}_z)$$

$$= \int_{x_B}^{x_A} E_x\, dx + \int_{y_B}^{y_A} E_y\, dy + \int_{z_B}^{z_A} E_z\, dz$$

Example

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

An electric field is given as: $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$

We wish to find the work done in moving a point charge of magnitude $Q = 2$ over the shorter arc of the circle given by $x^2 + y^2 = 1 \quad z = 1$

Giving that the initial point is $B(1, 0, 1)$ and the final point is $A(0.8, 0.6, 1)$.

$$W = -Q \int_B^A \mathbf{E} \cdot d\mathbf{L}$$

$$= -2 \int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$

$$= -2 \int_1^{0.8} y\,dx - 2 \int_0^{0.6} x\,dy - 4 \int_1^1 dz$$

This is the basic setup, in which the path has not yet been specified.

Example (continued)

We now have $W = -2\int_{1}^{0.8} y\,dx - 2\int_{0}^{0.6} x\,dy - 4\int_{1}^{1} dz$

and we need to include the y dependence on x in the first integral, and the x dependence on y in the second integral:

Using the given equation for the circ $x^2 + y^2 = 1 \quad z = 1$ we rewrite the integrals:

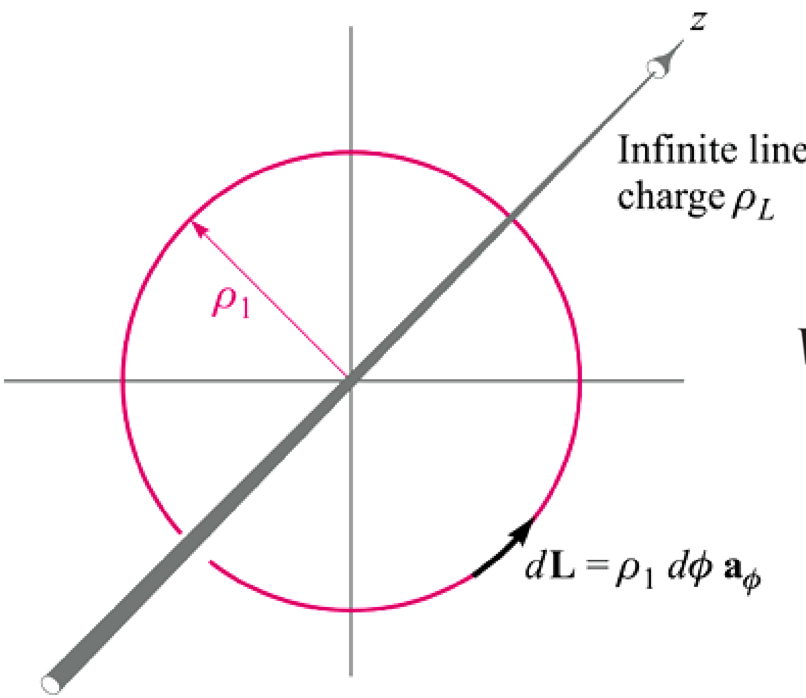
$$
\begin{aligned}
W &= -2\int_{1}^{0.8} \sqrt{1-x^2}\,dx - 2\int_{0}^{0.6} \sqrt{1-y^2}\,dy - 0 \\
&= -\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_{1}^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1} y\right]_{0}^{0.6} \\
&= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0) \\
&= -0.96\,\text{J}
\end{aligned}
$$

Evaluating Work within a Line Charge Field

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

In this example, the differential element $d\mathbf{L}$ is chosen in cylindrical coordinates, and the Circular path selected demands that $d\rho$ and dz be zero, so $d\mathbf{L} = \rho_1\, d\varphi\, \mathbf{a}\varphi$.

Therefore, the work in moving charge Q in a circular path around a line charge is found:



(a)

where $\mathbf{E} = E_\rho \mathbf{a}_\rho = \dfrac{\rho_L}{2\pi\epsilon_0\rho}\mathbf{a}_\rho$

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\, d\phi\, \mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\, \mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

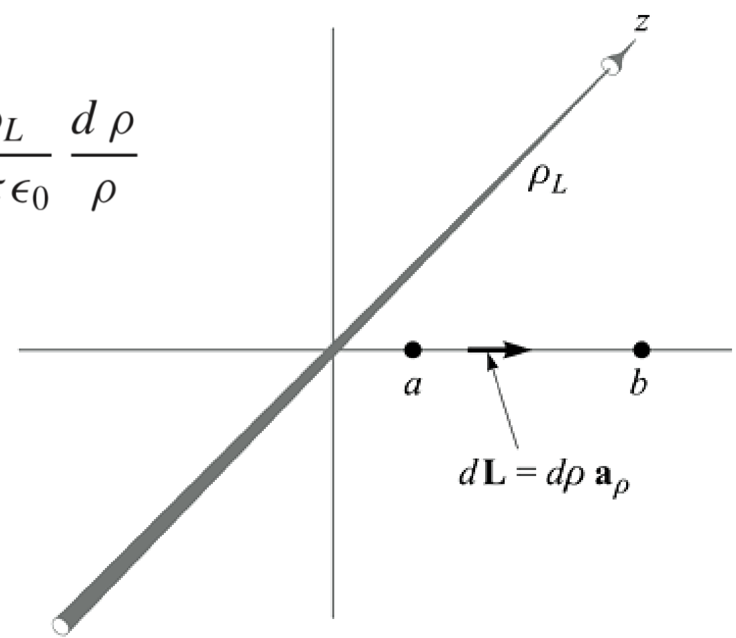
as expected!

Note that the path is always **<u>perpendicular</u>** to the electric field intensity, or the force on the charge is always exerted at right angles to the direction in which we are moving it.

Radial Motion Near a Line Charge

Instead, we now move charge Q along a radial line near the same line charge:

$$W = -Q \int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0 \rho} \mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho = -Q \int_a^b \frac{\rho_L}{2\pi\epsilon_0} \frac{d\rho}{\rho}$$



so that finally: $W = -\dfrac{Q\rho_L}{2\pi\epsilon_0} \ln \dfrac{b}{a}$

Differential Path Lengths in the Three Coordinate Systems

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(rectangular)}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\;d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

Definition of Potential Difference

We now have the work done in moving charge Q from initial to final positions. **This is the potential energy gained by the charge as a result of this position change.**

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

The *potential difference* is defined as

the work done (or potential energy gained) *per unit charge*,

expressed in units of Joules/Coulomb, or *volts*:

$$\text{Potential Difference} = \frac{W}{Q} = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \quad \text{Volts}$$

Finally:

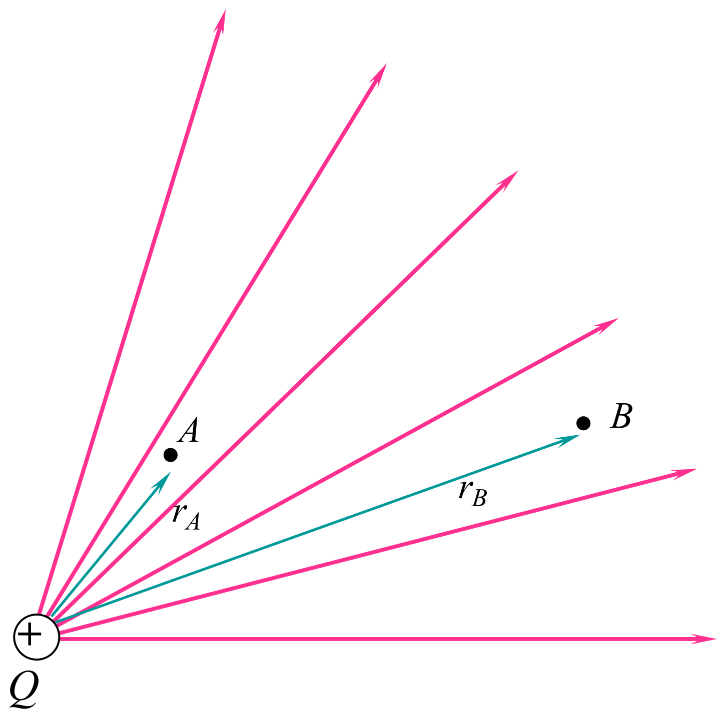
$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

Reference point to measure Potential Difference

- **It is often convenient to speak of the potential, or absolute potential, of a point, rather than the potential difference between two points,**
- This means only that we agree **to measure every potential difference with respect to a specified reference point that we consider to have zero potential.**
  - The most universal zero reference point is "**ground**," (i.e. the potential of the surface region of the earth itself).
  - Or "**infinity**" usually assumed for approximating a physical situation in which the earth is relatively far removed from the region in which we are interested (charges on airplane wings, atom potential, ect.)
  - A **cylindrical surface** of some definite radius may occasionally be used as a zero reference.

Potential Difference in a Point Charge Field

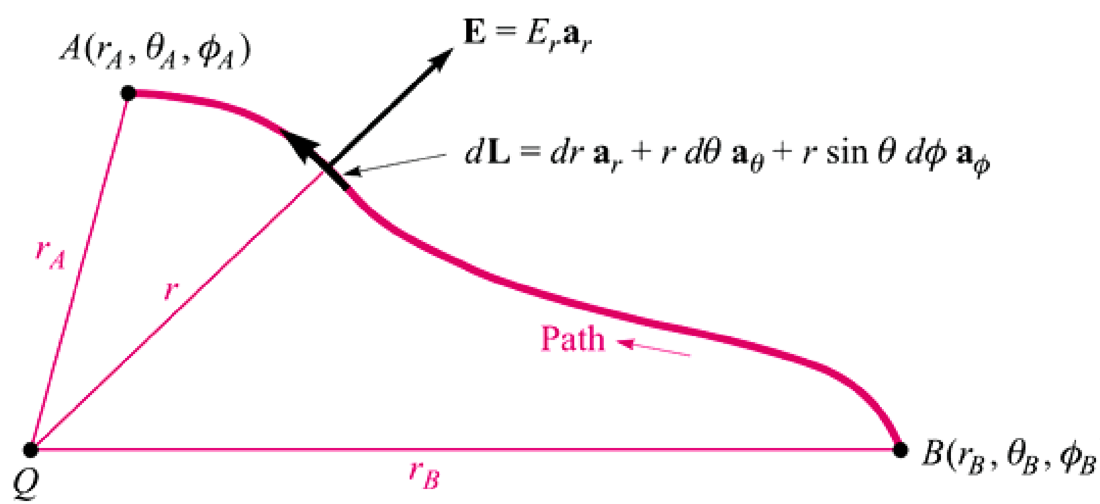
In this exercise, we evaluate the work done in moving a unit positive charge from point B to point A, within the field associated with point charge Q .



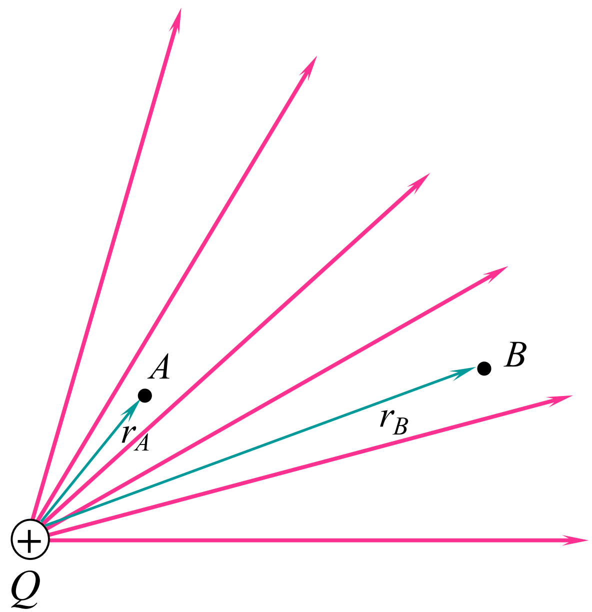
where $$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

and where in general:

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$



Potential Difference in a Point Charge Field



To complete the problem:

we use

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

along with:

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$

to obtain:

$$V_{AB} = -\int_B^A \mathbf{E}\cdot d\mathbf{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

- An inspection of the form of the **potential field of a point charge** shows that it is **an inverse-distance** field,
- whereas the electric **field intensity** was found to be **an inverse-square-law function.**

The potential field of a system of charges: conservative field

- **A field is conservative if its line integral between any two points is independent of the path chosen.**
- **Most fields in nature are conservative (as this also implies conservation of energy; e.g., the Earth's gravitational field).**
- **Another property of a conservative field is that its *closed path* line integral is zero: (true for static fields ONLY)**

$$\oint \mathbf{F} \cdot d\mathbf{L} = 0$$

The Potential Field of a Point Charge

We just found the difference in potential between two positions in a point charge field:

$$V_{AB} = V_A - V_B = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

We could perform the same calculation by specifying the starting point at infinity, and the ending point at some general radius, r:

$$V_{r\infty} = V_r - V_\infty = -\int_\infty^r \mathbf{E}\cdot d\mathbf{L} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{Q}{4\pi\epsilon_0 r}$$

This result is a *potential function* or *potential field*, which specifies potential at any position within the defined space, and with the potential at infinity (the reference value) equal to zero.

In practice, we can "bias" this function any way we want (or need) to, by an additive constant, C_1:

$$V = \frac{Q}{4\pi\epsilon_0 r} + C_1$$

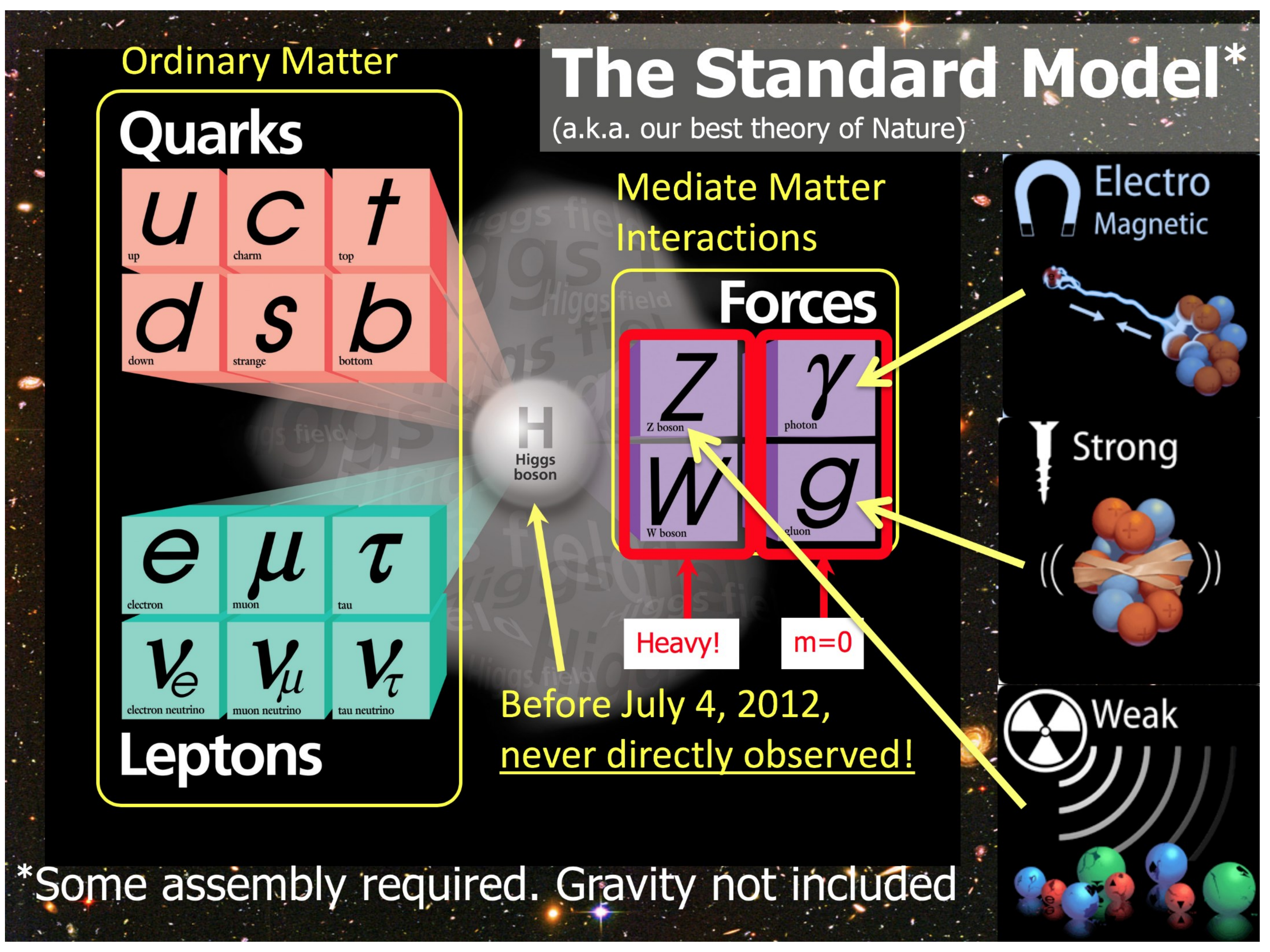
Potential Difference in a Point Charge Field

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

This expression defines the potential at any point distant r from a point charge Q at the origin, the potential at infinite radius being taken as the zero reference.

Accordingly, the physical interpretation of the Potential is that:

$Q/4\pi \epsilon_o r$ joules of work must be done in carrying a unit charge from infinity to any point r meters from the charge Q.

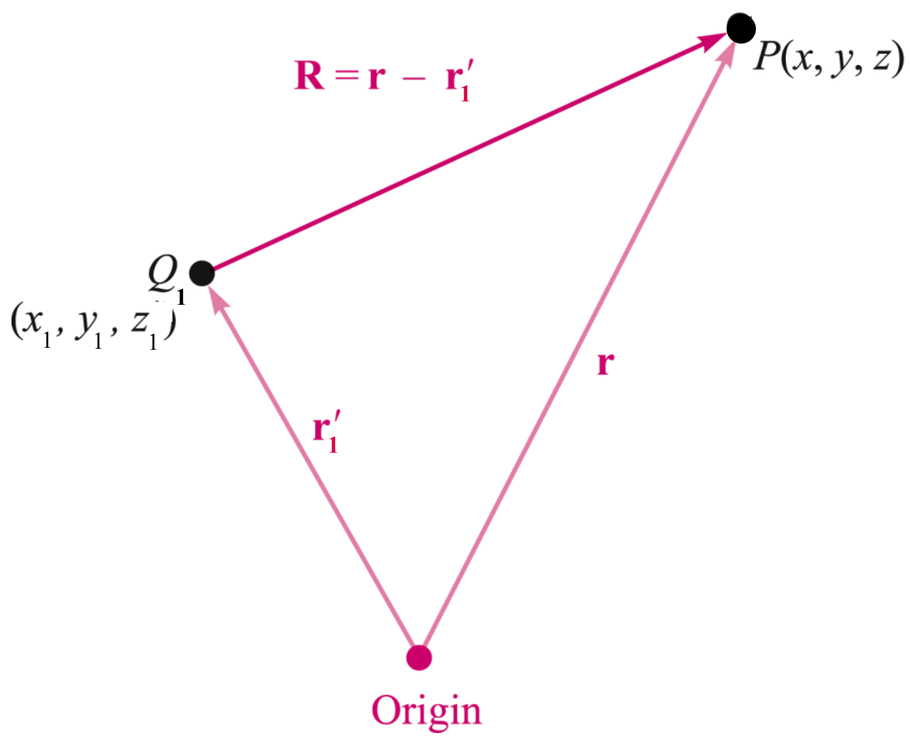


The Standard Model*
(a.k.a. our best theory of Nature)
Ordinary Matter
Quarks
u up
c charm
t top
d down
s strange
b bottom
e electron
μ muon
τ tau
νe electron neutrino
νμ muon neutrino
ντ tau neutrino
Leptons
H Higgs boson
Mediate Matter Interactions
Forces
Z Z boson
γ photon
W W boson
g gluon
Heavy!
m=0
Electro Magnetic
Strong
Weak
Before July 4, 2012,
never directly observed!
*Some assembly required. Gravity not included

Potential Field of a Point Charge Off-Origin

The setup here is the same as what we used in writing the electric field of an off-origin point charge.

$$V_P(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|}$$



Potential Field Arising From Two or More Point Charges

Introduce a second point charge, and the two scalar potentials simply add:

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|}$$

For n charges, the process continues:

$$V(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|}$$

Potential Associated with Continuous Charge Distributions

If each point charge is now represented as a small element of a continuous volume charge distribution $\rho_v \Delta v$, then

$$V(\mathbf{r}) = \frac{\rho_\nu(\mathbf{r}_1)\Delta\nu_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_\nu(\mathbf{r}_2)\Delta\nu_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \cdots + \frac{\rho_\nu(\mathbf{r}_n)\Delta\nu_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

As we allow the number of elements to become infinite, we obtain the integral expression:

$$V(\mathbf{r}) = \int_{\mathrm{vol}} \frac{\rho_\nu(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Potential Functions Associated with Line, Surface, and Volume Charge Distributions

Line Charge: $$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\, dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Surface Charge: $$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\, dS'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Volume Charge: $$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\, dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

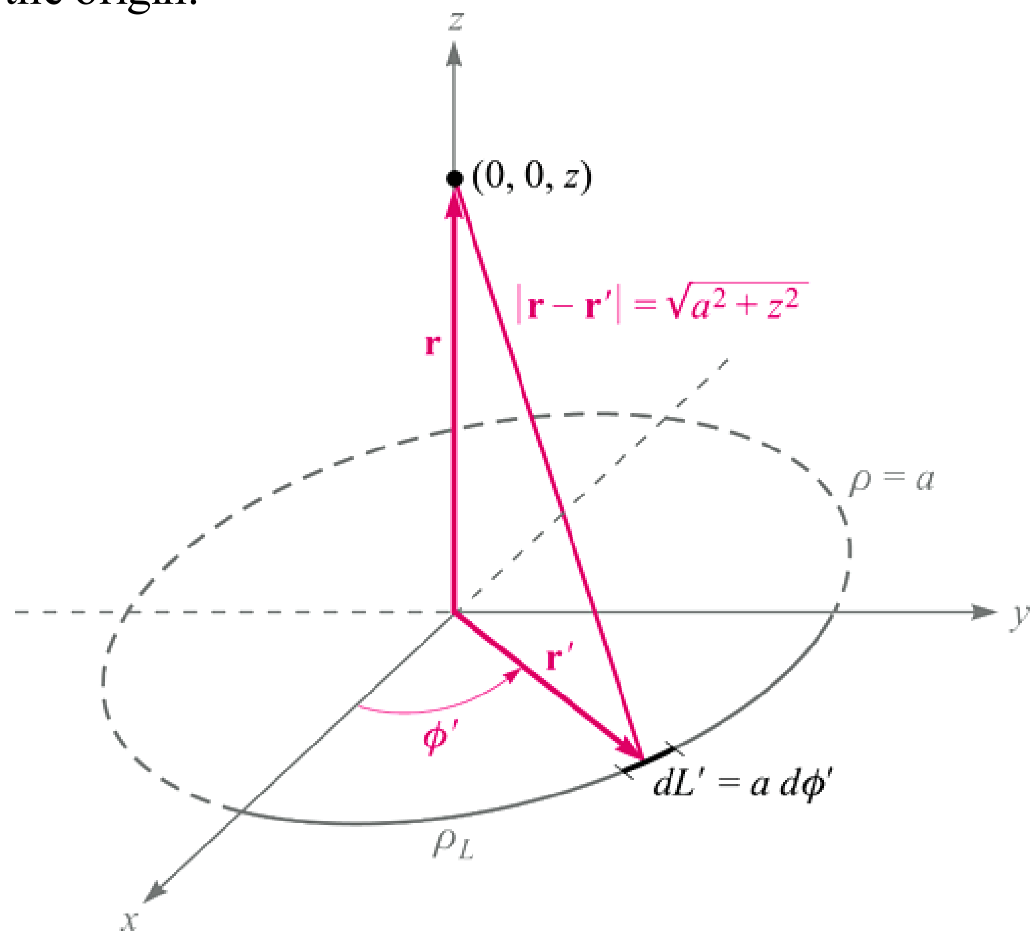
- **The potential is inverse distance, and the electric field intensity, inverse square.**
- **The potential is scalar, but the electric field is a vector.**

Compare to our earlier expression for electric field --- generally a more difficult integral to evaluate:

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\, dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

Example

The problem is to find the potential anywhere on the *z* axis arising from a circular ring of charge in the *x*-*y* plane, centered at the origin.



We use:

$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\,dL'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

with $dL' = a d\phi'$

$$\mathbf{r} = z\mathbf{a}_z$$

$$\mathbf{r}' = a\mathbf{a}_\rho$$

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{a^2+z^2}$$

Example (continued)

So now $V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\, dL'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$

becomes:

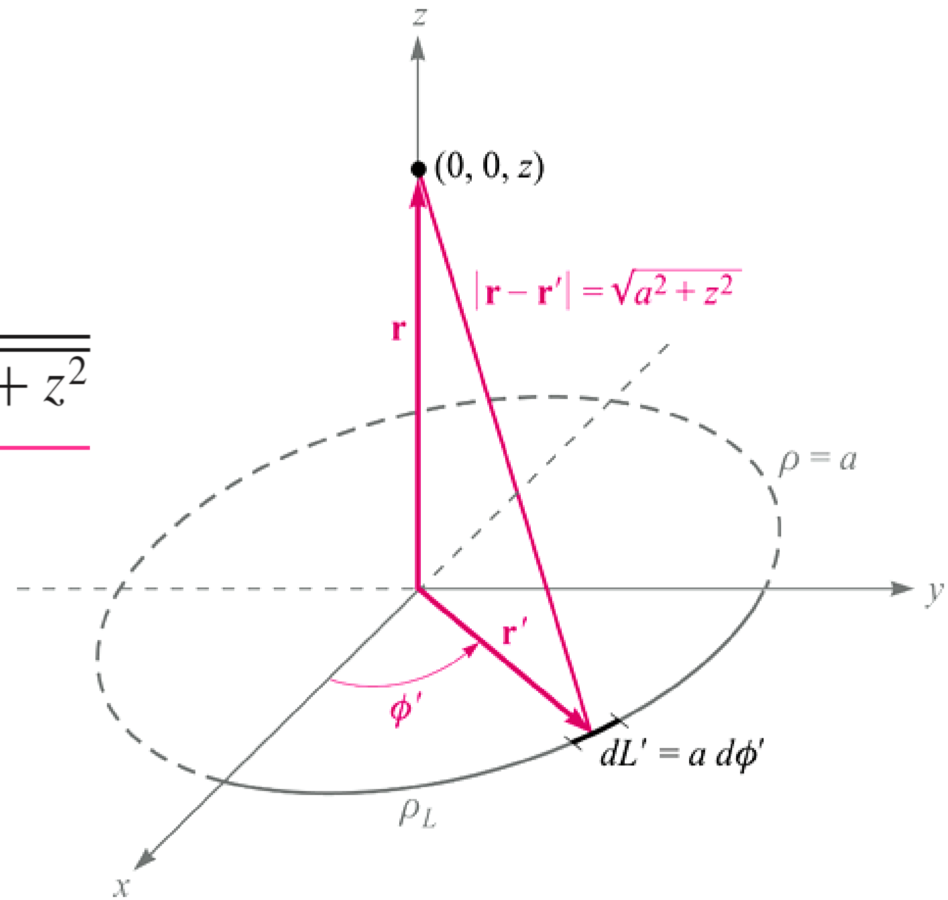
$$V = \int_0^{2\pi} \frac{\rho_L a\, d\phi'}{4\pi\epsilon_0\sqrt{a^2 + z^2}} = \frac{\rho_L a}{2\epsilon_0\sqrt{a^2 + z^2}}$$

where $dL' = a d\phi'$

$$\mathbf{r} = z\mathbf{a}_z$$

$$\mathbf{r}' = a\mathbf{a}_\rho$$

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{a^2 + z^2}$$



Change in Voltage over an Incremental Distance

The change in potential occurring over distance ΔL depends on the **angle** between this vector and the electric field; i.e., the projection of the field along the path:

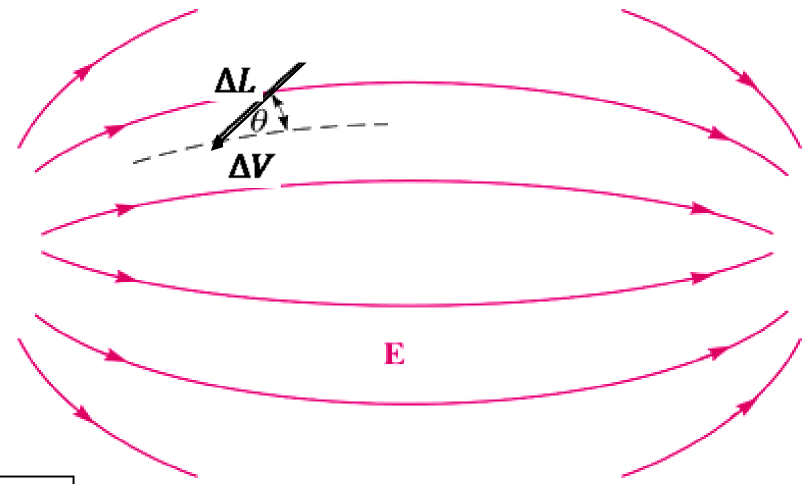
$$\Delta V \doteq -\mathbf{E} \cdot \Delta \mathbf{L}$$

or

$$\Delta V \doteq -E \, \Delta L \cos\theta$$

from which: $$\frac{dV}{dL} = -E \cos\theta$$

whose maximum value is: $$\left. \frac{dV}{dL} \right|_{max} = E$$



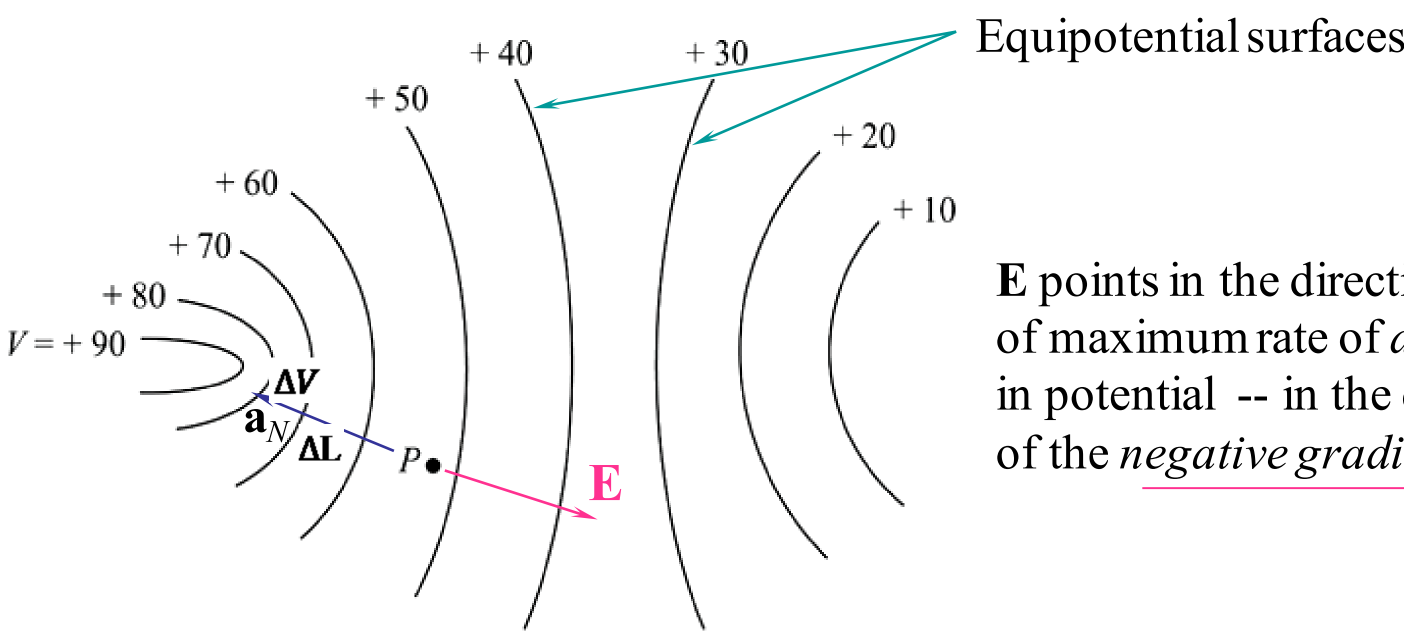
1. The magnitude of the electric field intensity is given by the maximum value of the **rate of change of potential with distance**.
2. This maximum value is obtained when the direction of E is **opposite** to the direction in which the potential is increasing the most rapidly.

Relation Between Potential and Electric Field

The maximum rate of *increase* in potential should occur in a direction ***exactly opposite*** the electric field:

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N$$

$\mathbf{a}_N$: unit vector normal to an equipotential surface and in the direction of *increasing* potential



E points in the direction of maximum rate of *decrease* in potential -- in the direction of the *negative gradient* of *V*.

Potential Gradient

We now have two methods of determining potential,

1. **one directly from the electric field intensity by means of a line integral, and**
2. **from the basic charge distribution itself by a volume integral.**

Neither method is very helpful in determining the fields in most practical problems, since neither the electric field intensity nor the charge distribution is very often known.

Therefore, Potential Gradient ∇V could be used to approximately determine the **magnitude and direction of the potential fields**, i.e. to measure the maximum space rate of change of a scalar quantity V and *the direction in which this maximum occurs.*

Electric Field in Terms of *V* in Rectangular Coordinates

The differential voltage change can be written as the sum of changes of *V* in the three coordinate directions:

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

We also know that: $dV = -\mathbf{E} \cdot d\mathbf{L} = -E_x\, dx - E_y\, dy - E_z\, dz$

We therefore identify:

$$\left\{ \begin{aligned} E_x &= -\frac{\partial V}{\partial x} \\ E_y &= -\frac{\partial V}{\partial y} \\ E_z &= -\frac{\partial V}{\partial z} \end{aligned} \right.$$

So that:

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

Electric Field as the Negative Gradient of the Potential Field

We now have the relation between **E** and *V*

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z\right)$$

This is obtained by using the del operator, $\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$ on V

A more compact relation therefore emerges, which is applicable to *static* electric fields:

$$\mathbf{E} = -\nabla V$$

E is equal to the *negative gradient* of *V*

The direction of the gradient is that of the maximum rate of *increase* in the scalar field, or normal to all equipotential surfaces.

Gradient of *V* in the Three Coordinate Systems

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(rectangular)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

Example

Given the potential field, $V = 2x^2y - 5z$, and a point $P(-4, 3, 6)$, we wish to find several numerical values at point P: the potential V, the electric field intensity **E**, the direction of **E**, the electric flux density **D**, and the volume charge density ρ_v.

Solution. The potential at $P(-4, 5, 6)$ is

$$V_P = 2(-4)^2(3) - 5(6) = 66 \text{ V}$$

Next, we may use the gradient operation to obtain the electric field intensity,

$$\mathbf{E} = -\nabla V = -4xy\mathbf{a}_x - 2x^2\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

The value of **E** at point P is

$$\mathbf{E}_P = 48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z \text{ V/m}$$

and

$$|\mathbf{E}_P| = \sqrt{48^2 + (-32)^2 + 5^2} = 57.9 \text{ V/m}$$

The direction of **E** at P is given by the unit vector

$$\mathbf{a}_{E,P} = (48\mathbf{a}_x - 32\mathbf{a}_y + 5\mathbf{a}_z)/57.9$$
$$= 0.829\mathbf{a}_x - 0.553\mathbf{a}_y + 0.086\mathbf{a}_z$$

Example

If we assume these fields exist in free space, then

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -35.4xy\,\mathbf{a}_x - 17.71x^2\,\mathbf{a}_y + 44.3\,\mathbf{a}_z \text{ pC/m}^3$$

Finally, we may use the divergence relationship to find the volume charge density that is the source of the given potential field,

$$\rho_v = \nabla \cdot \mathbf{D} = -35.4y \text{ pC/m}^3$$

At P, $\rho_v = -106.2 \text{ pC/m}^3$.

Exercises

4.1. The value of $\mathbf{E}$ at $P(\rho = 2, \phi = 40°, z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a 20 μC charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E} \cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6 \times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20 \times 10^{-6}\,\text{C})(100\,\text{V/m})(6 \times 10^{-6}\,\text{m}) = -12 \times 10^{-9}\text{ J} = \underline{-12\,\text{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6 \times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20 \times 10^{-6})(-200)(6 \times 10^{-6}) = 2.4 \times 10^{-8}\,\text{J} = \underline{24\text{ nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6 \times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20 \times 10^{-6})(300)(6 \times 10^{-6}) = -3.6 \times 10^{-8}\,\text{J} = \underline{-36\text{ nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6 \times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(rectangular)}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

Thus

$$dW = -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times 10^{-6})$$
$$= \underline{-44.9\,\text{nJ}}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2+3^2+4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$dW = -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times 10^{-6})$$
$$= -(20\times 10^{-6})\,[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y)$$
$$+\,222.9]\,(6\times 10^{-6})$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40^\circ) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40^\circ) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40^\circ) = -0.643$. Substituting these results in

$$dW = -(20\times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times 10^{-6}) = \underline{-41.8\,\text{nJ}}$$

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	0

4.4. An electric field in free space is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work done in moving a 1μC charge through this field

a) from (1,1,1) to (0,0,0): The work will be

$$W = -q\int \mathbf{E}\cdot d\mathbf{L} = -10^{-6}\left[\int_1^0 x\,dx + \int_1^0 y\,dy + \int_1^0 z\,dz\right] \text{J} = \underline{1.5\,\mu\text{J}}$$

b) from ($\rho = 2$, $\phi = 0$) to ($\rho = 2$, $\phi = 90°$): The path involves changing ϕ with ρ and z fixed, and therefore $d\mathbf{L} = \rho\, d\phi\, \mathbf{a}_\phi$. We set up the integral for the work as

$$W = -10^{-6}\int_0^{\pi/2} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)\cdot \rho\, d\phi\, \mathbf{a}_\phi$$

where $\rho = 2$, $\mathbf{a}_x\cdot\mathbf{a}_\phi = -\sin\phi$, $\mathbf{a}_y\cdot\mathbf{a}_\phi = \cos\phi$, and $\mathbf{a}_z\cdot\mathbf{a}_\phi = 0$. Also, $x = 2\cos\phi$ and $y = 2\sin\phi$. Substitute all of these to get

$$W = -10^{-6}\int_0^{\pi/2} \left[-(2)^2\cos\phi\sin\phi + (2)^2\cos\phi\sin\phi\right] d\phi = \underline{0}$$

Given that the field is conservative (and so work is path-independent), can you see a much easier way to obtain this result?

$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$ (rectangular)

$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\mathbf{a}_\phi + dz\,\mathbf{a}_z$ (cylindrical)

$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$ (spherical)

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	0

c) from $(r = 10,\ \theta = \theta_0)$ to $(r = 10,\ \theta = \theta_0 + 180^\circ)$: In this case, we are moving only in the $\mathbf{a}_\theta$ direction. The work is set up as

$$W = -10^{-6} \int_{\theta_0}^{\theta_0+\pi} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) \cdot r\, d\theta\, \mathbf{a}_\theta$$

Now, substitute the following relations: $r = 10$, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, $\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos\theta\cos\phi$, $\mathbf{a}_y \cdot \mathbf{a}_\theta = \cos\theta \sin\phi$, and $\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin\theta$. Obtain

$$W = -10^{-6} \int_{\theta_0}^{\theta_0+\pi} (10)^2 \left[\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \cos\theta\sin\theta\right] d\theta = \underline{0}$$

where we use $\cos^2\phi + \sin^2\phi = 1$.

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

4.6. An electric field in free space is given as $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z$. Given $V(1,1,1) = 10$ V. Determine $V(3,3,3)$. The potential difference is expressed as

$$V(3,3,3) - V(1,1,1) = -\int_{1,1,1}^{3,3,3} (x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$

$$= -\left[\int_1^3 x\,dx + \int_1^3 4z\,dy + \int_1^3 4y\,dz\right]$$

We choose the following path: 1) move along x from 1 to 3; 2) move along y from 1 to 3, holding x at 3 and z at 1; 3) move along z from 1 to 3, holding x at 3 and y at 3. The integrals become:

$$V(3,3,3) - V(1,1,1) = -\left[\int_1^3 x\,dx + \int_1^3 4(1)\,dy + \int_1^3 4(3)\,dz\right] = -36$$

So

$$V(3,3,3) = -36 + V(1,1,1) = -36 + 10 = \underline{-26}\text{ V}$$

$$V_{AB} = V_A - V_B = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

4.14. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$, find the potential difference between the points

a) (2,-2,-1) and (0,0,0): We choose a path along which motion occurs in one coordinate direction at a time. Starting at the origin, first move along x from 0 to 2, where $y = 0$; then along y from 0 to -2, where x is 2; then along z from 0 to -1. The setup is

$$V_b - V_a = -\int_0^2 (y+1)\Big|_{y=0} dx \; - \int_0^{-2} (x-1)\Big|_{x=2} dy \; - \int_0^{-1} 2\, dz = \underline{2}$$

b) (3,2,-1) and (-2,-3,4): Following similar reasoning,

$$V_b - V_a = -\int_{-2}^{3} (y+1)\Big|_{y=-3} dx \; - \int_{-3}^{2} (x-1)\Big|_{x=3} dy \; - \int_{4}^{-1} 2\, dz = \underline{10}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) **E**: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4}\,\mathrm{V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0\mathbf{E}$, we find the charge density through

$$\rho_v\Big|_{.5} = [\nabla \cdot \mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673\,\mathrm{pC/m^3}}$$

c) the total charge lying within the closed surface $\rho = .6$, $0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part *a*, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521\,\mathrm{pC/m^2}$. Thus $Q = -2\pi(.6)(1)521 \times 10^{-12}\,\mathrm{C} = \underline{-1.96\,\mathrm{nC}}$.

$$D_S = D_\rho = \frac{Q}{2\pi\rho L}$$